

# Objectives

- ◆ To develop methods for determining the moment of inertia for an area.
- ◆ To understand the use of the parallel axis theorem.

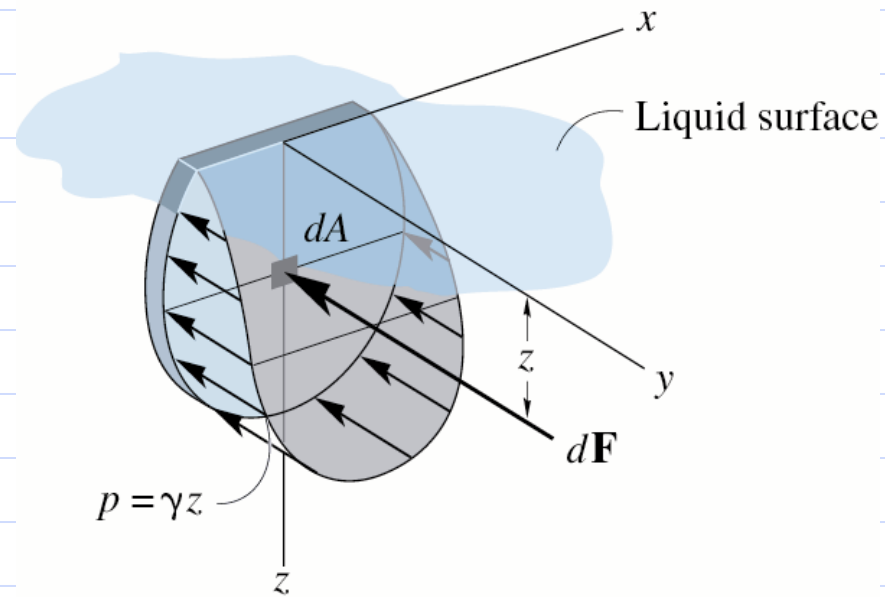


Figure 10.01

Called “second moment” of  
the area or the moment of  
inertia

$$\int x^2 dA$$

or

$$\int y^2 dA$$

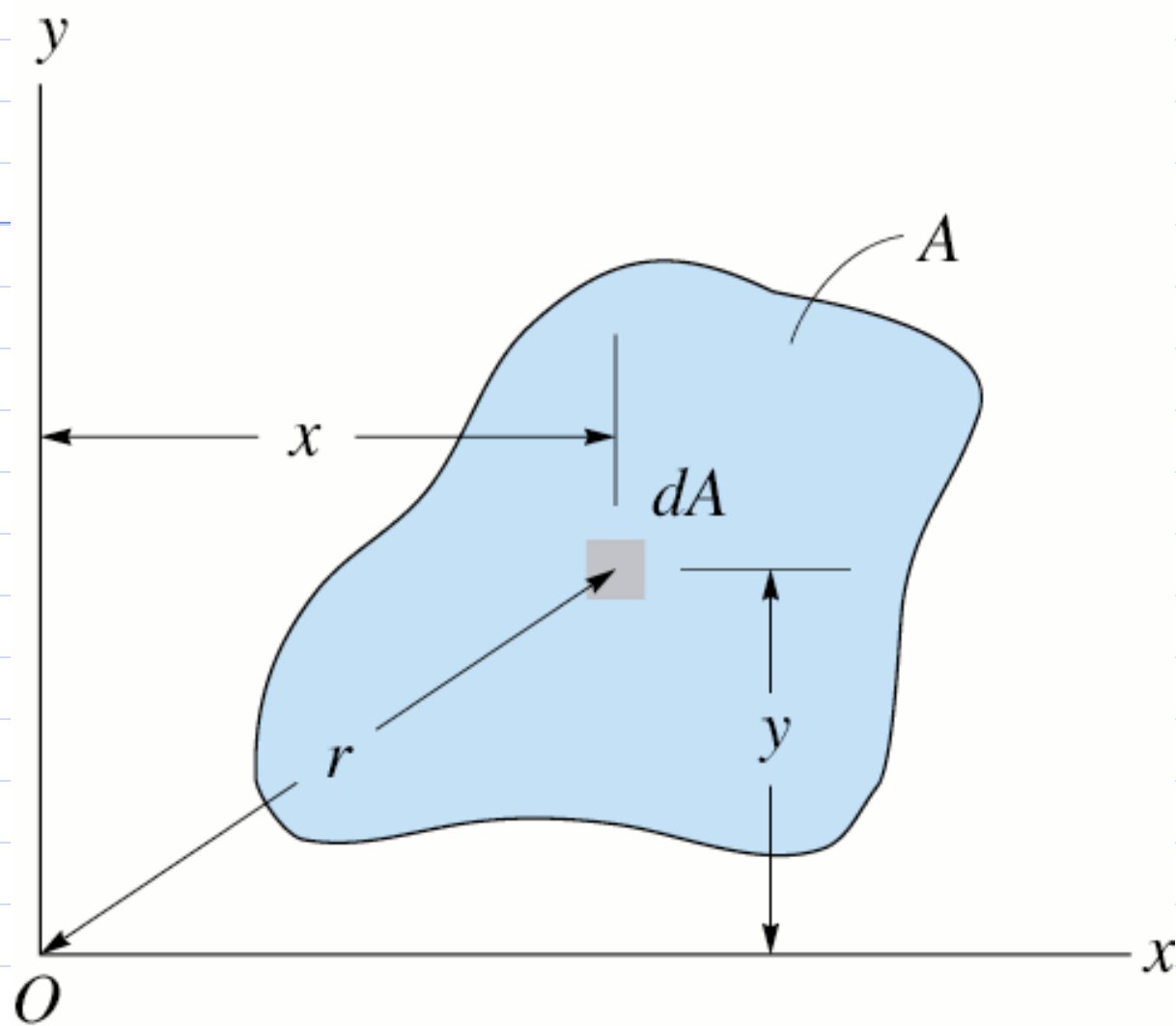


Figure 10.02

$$\mathbf{I_x = \int y^2 dA}$$

$$\mathbf{I_y = \int x^2 dA}$$

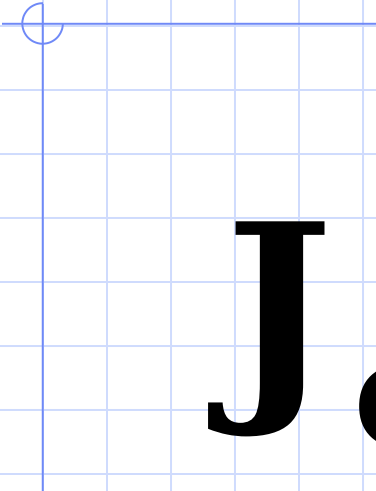
**and**


$$\mathbf{J_o = \int r^2 dA = I_x + I_y}$$

# Polar Moment of Inertia

$$J_o = \int r^2 dA = I_x + I_y$$

$$r^2 = x^2 + y^2$$


$$\mathbf{J_o}, \mathbf{I_x}, \mathbf{I_y} > 0$$



**Units:**  
**(length)<sup>4</sup>**

**m<sup>4</sup>**

**mm<sup>4</sup>**

**ft<sup>4</sup>**

**in<sup>4</sup>**



# Parallel Axis Theorem

$$I_x = \int y^2 dA$$

$$I_x = \int (y' + d_y)^2 dA$$

$$I_x = \int (y'^2 + 2d_y y' + d_y^2) dA$$

$$I_x = \int y'^2 dA + 2d_y \int y' dA + d_y^2 \int dA$$

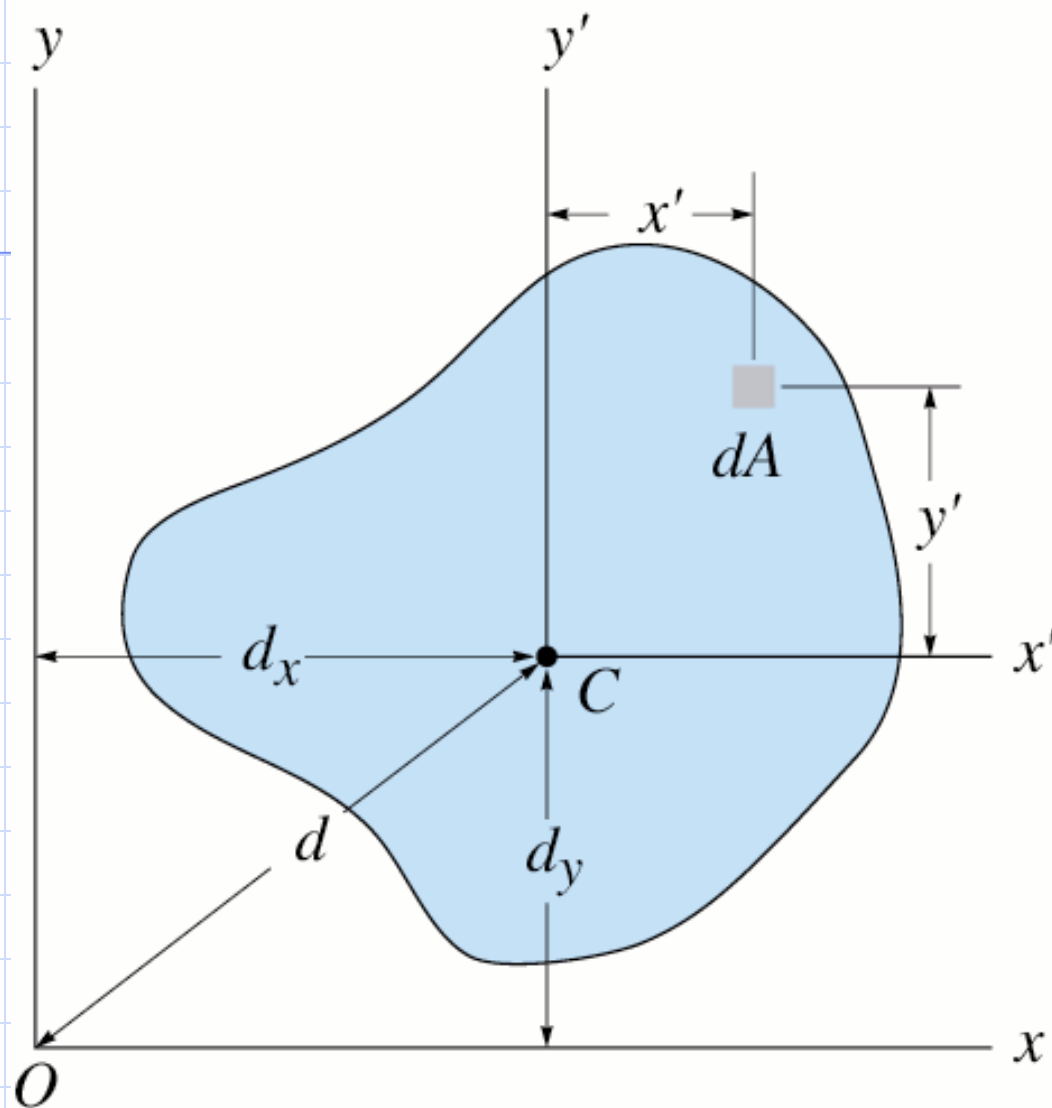


Figure 10.03

# Parallel Axis Theorem

$$I_x = \int y'^2 dA + 2d_y \int y' dA + d_y^2 \int dA$$

$$\int y' dA = 0 \quad \textit{centroid}$$

$$I_x = \bar{I}_{x'} + A d_y^2$$

# Parallel Axis

## Theorem:

$$I_x = \bar{I}_x' + Ad_y^2$$

$$I_y = \bar{I}_y' + Ad_x^2$$

$$J_o = \bar{J}_c + Ad^2$$

# Parallel Axis Theorem:

*The moment of inertia of an area about an axis is equal to the moment of inertia of the area about a parallel axis passing through the centroid plus the product of the area and the square of the perpendicular distance*

# Radius of Gyration of an Area

$$k_x = \sqrt{\frac{I_x}{A}}$$

$$k_y = \sqrt{\frac{I_y}{A}}$$

$$k_o = \sqrt{\frac{J_o}{A}}$$

# Procedure for Analysis

## Case I

- ◆ Specify a differential element  $dA$  with length parallel to axis.
- ◆ Apply appropriate integration formula.

$$I_y = \int x^2 dA$$

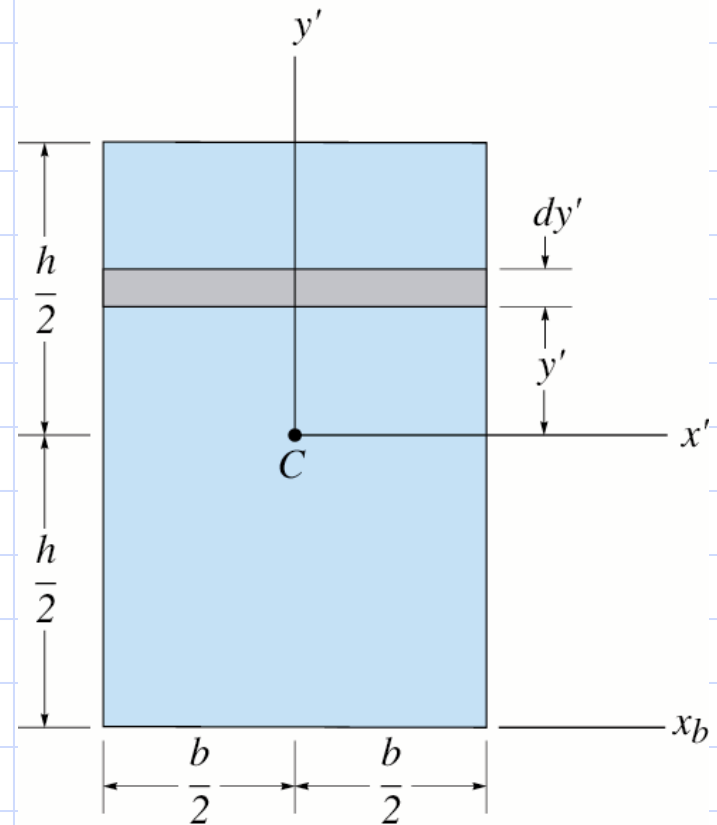


Figure 10.05

# Procedure for Analysis

## Case II

- ◆ Specify a differential element  $dA$  with length perpendicular to axis.
- ◆ Calculate  $I$  about centroidal axis and apply parallel axis theorem.

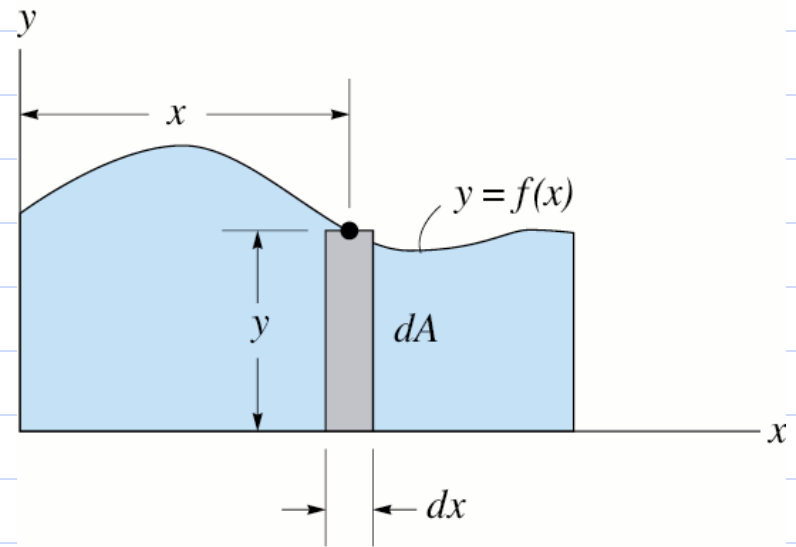
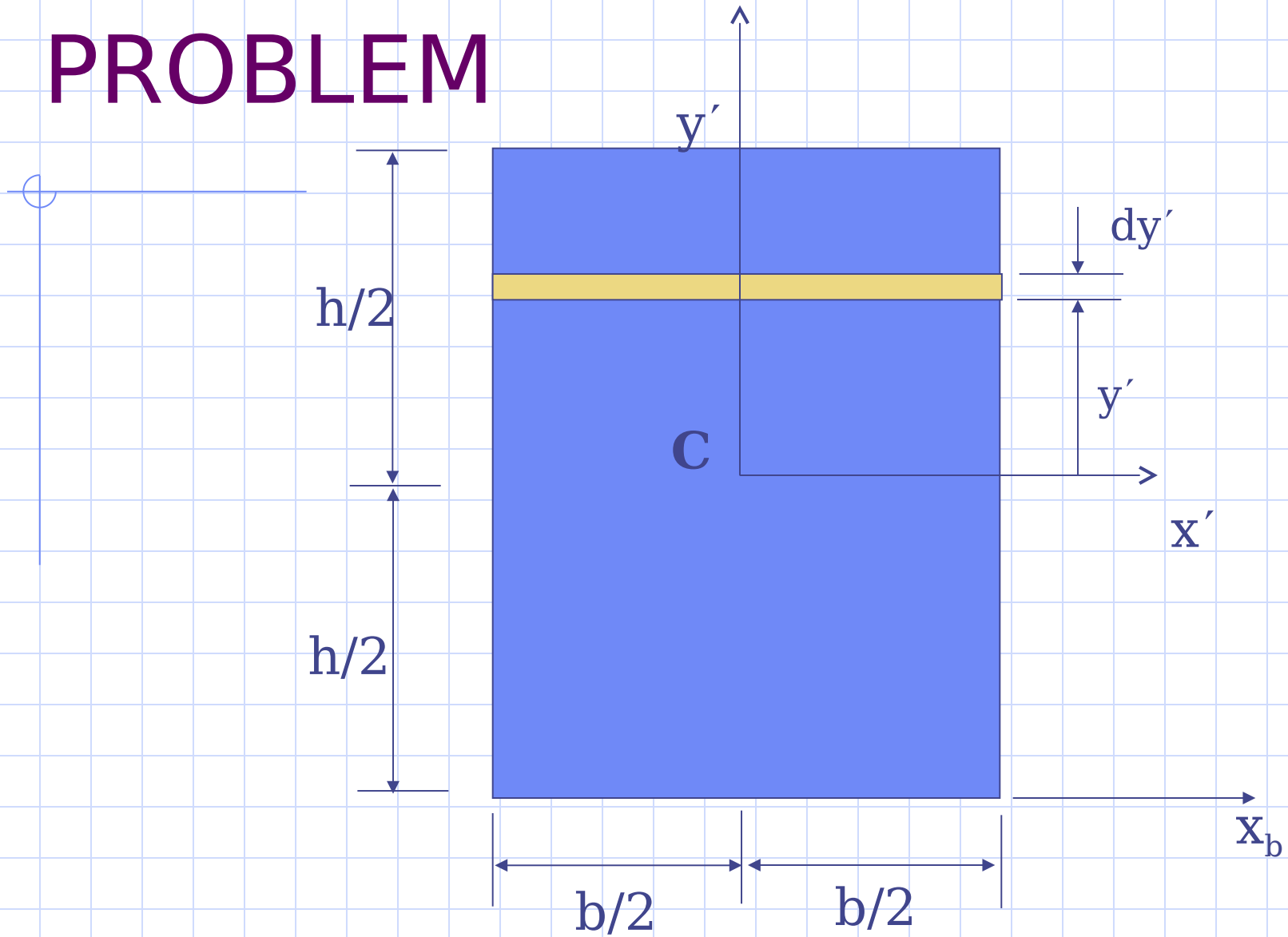


Figure 10.04



# PROBLEM



# Through Centroidal Axis:

$$\bar{I}_{x'} = \int_A y'^2 dA = \int_{-h/2}^{+h/2} y'^2 b dy'$$

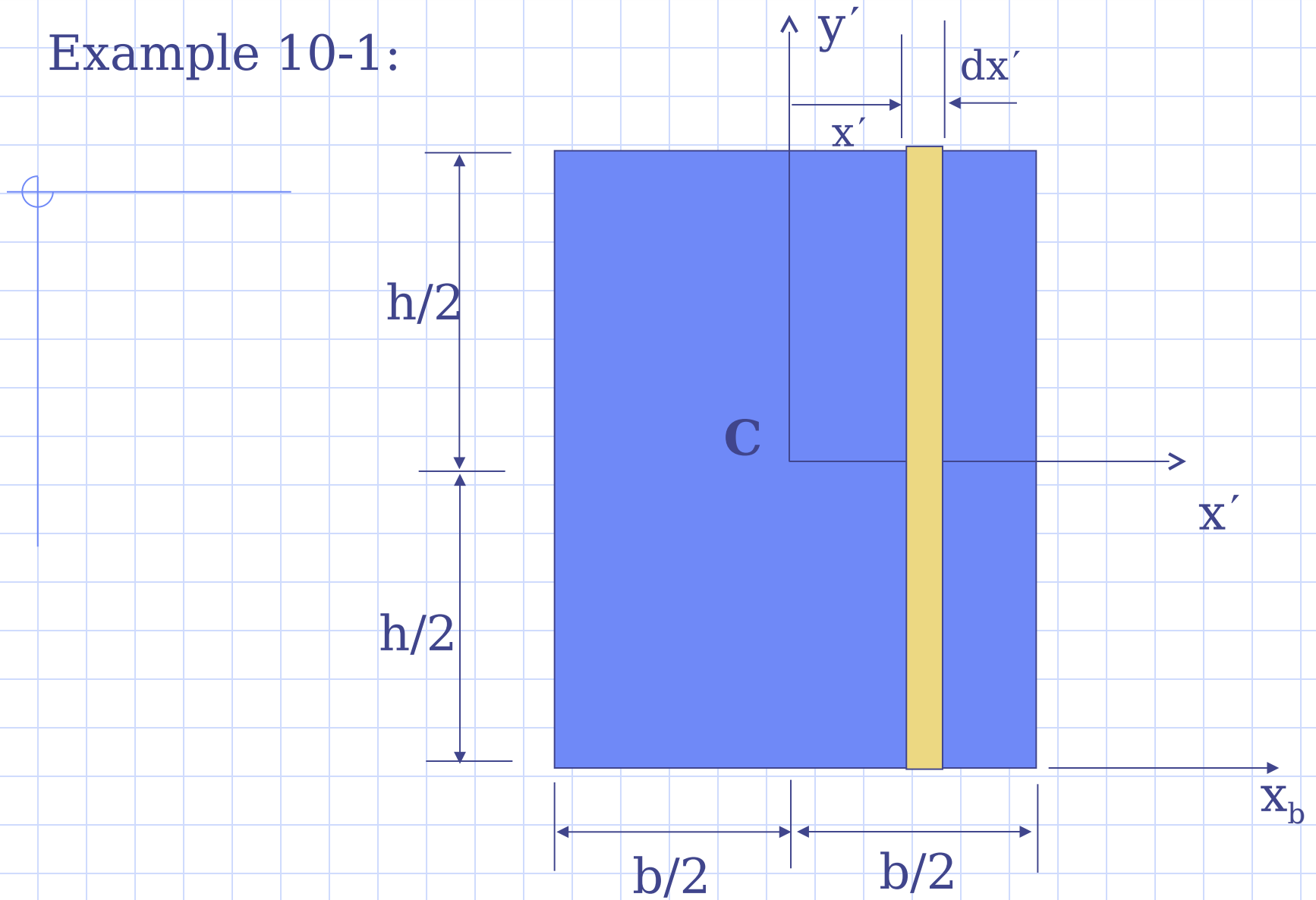
$$\bar{I}_{x'} = b \int_{-h/2}^{+h/2} y'^2 dy' = \frac{1}{12} b h^3$$

# through Axis Passing Thorough Base

$$\bar{\bar{\mathbf{I}}}_{\mathbf{x}_b} = \bar{\bar{\mathbf{I}}}_{\mathbf{x}'} + A \mathbf{d}_y^2$$

$$\bar{\bar{\mathbf{I}}}_{\mathbf{x}_b} = \frac{1}{12} \mathbf{b} \mathbf{h}^3 + \mathbf{b} \mathbf{h} \left( \frac{\mathbf{h}}{2} \right)^2 = \frac{1}{3} \mathbf{b} \mathbf{h}^3$$

# Example 10-1:



$$\bar{I}_{y'} = \int_A x'^2 dA = \int_{-b/2}^{+b/2} x'^2 h dx'$$

$$\bar{I}_{y'} = h \int_{-b/2}^{+b/2} x'^2 dx' = \frac{1}{12} hb^3$$

$$J_c = \bar{I}_{x'} + \bar{I}_{y'} = \frac{1}{12} (bh^3 + hb^3)$$

# PROBLEM

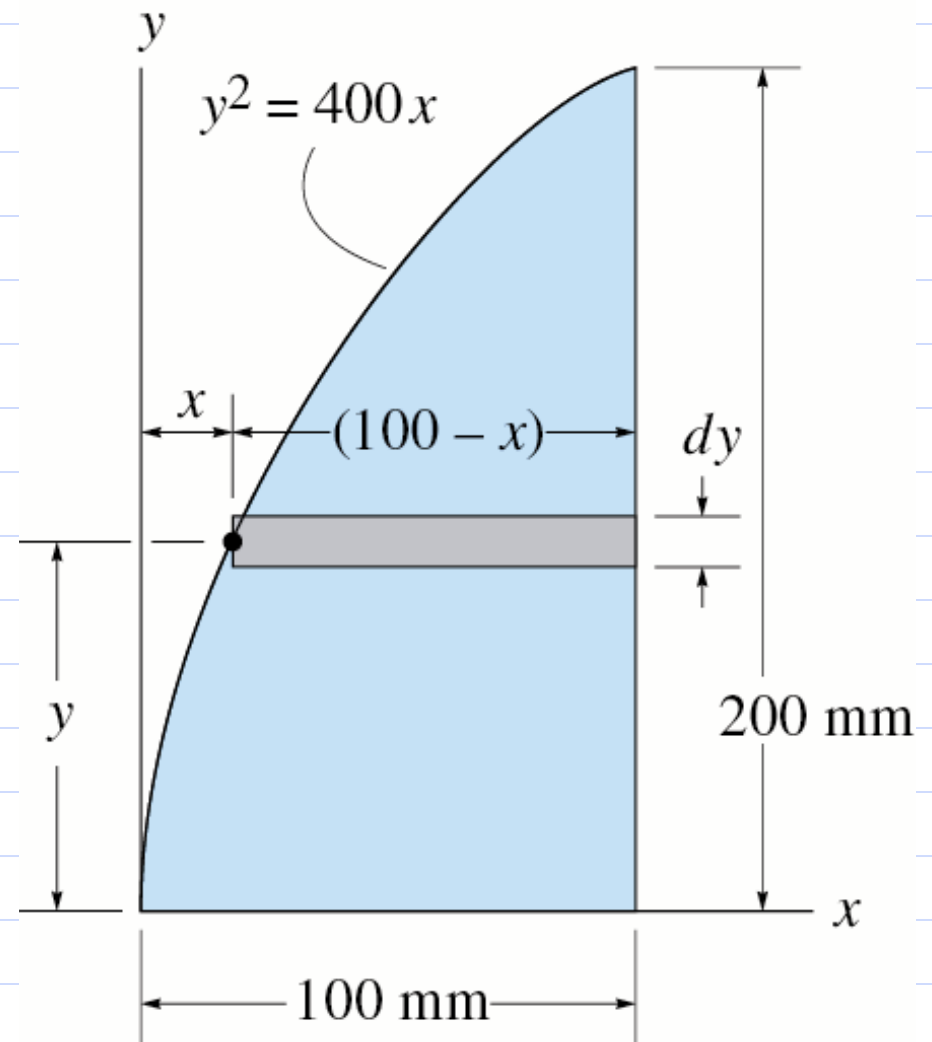


Figure 10.06(a)

$$I_x = \int_A y^2 dA = \int_A y^2 (100 \, x) dy$$

$$I_x = \int_0^{200} y^2 \left( 100 - \frac{y^2}{400} \right) dy$$

$$I_x = 100 \int_0^{200} y^2 dy - \frac{1}{400} \int_0^{200} y^4 dy$$

$$I_x = 107 \times 10^6 \, \text{mm}^4$$

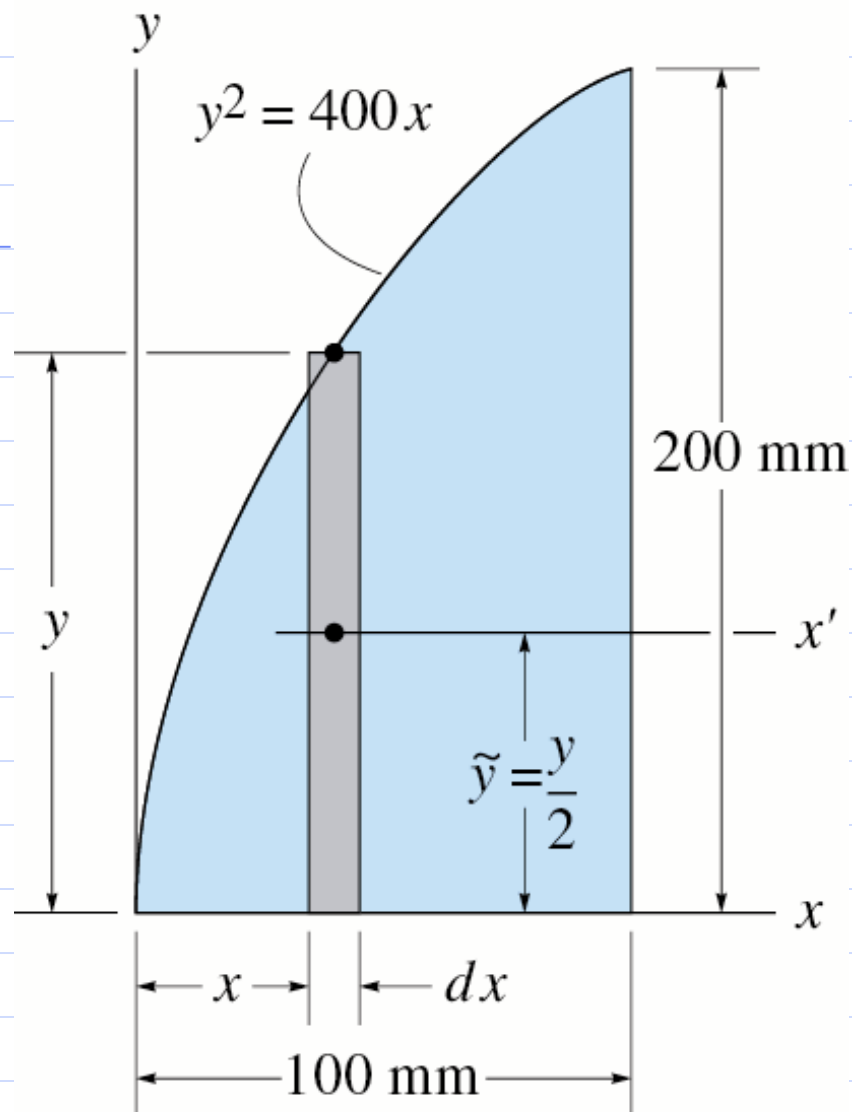


Figure 10.06(b)



$$\bar{I}_{x'} = \frac{1}{12}bh^3$$

$$d\bar{I}_{x'} = \frac{1}{12}dxy^3$$

$$dI_x = d\bar{I}_{x'} + dA\tilde{y}^2$$

$$dI_x = \frac{1}{12}dxy^3 + ydxy^2$$

$$dI_x = \frac{1}{12}dxy^3 + ydx\left(\frac{y}{2}\right)^2$$

$$dI_x = \frac{1}{3}y^3dx$$

$$dI_x = \frac{1}{3} y^3 dx$$

$$I_x = \int_A dI_x = \int_A \frac{1}{3} y^3 dx$$

$$I_x = \int_0^{100} \frac{1}{3} (400)^{\frac{3}{2}} dx$$

$$I_x = 107 \times 10^6 \text{ mm}^4$$

# PROBLEM

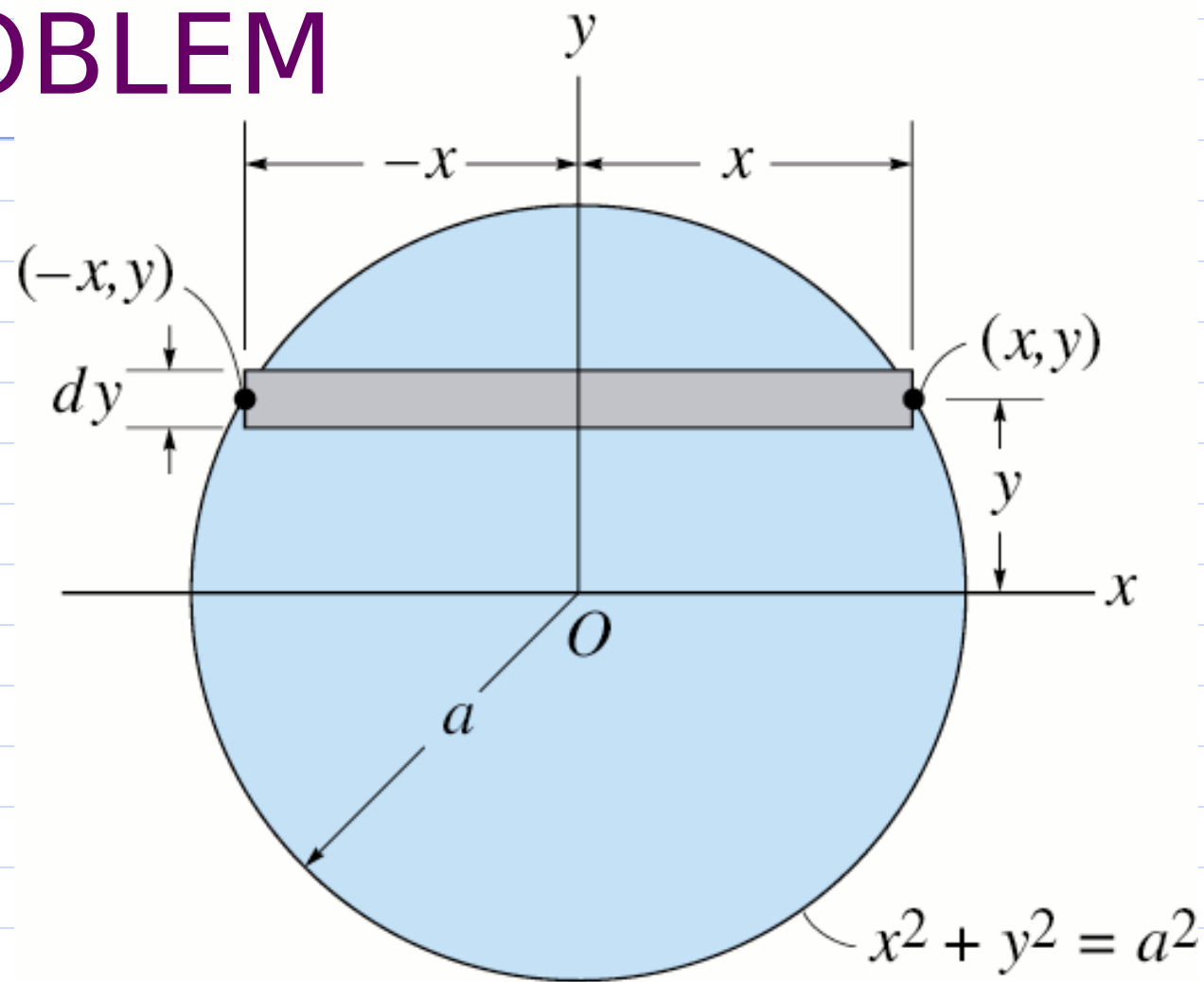
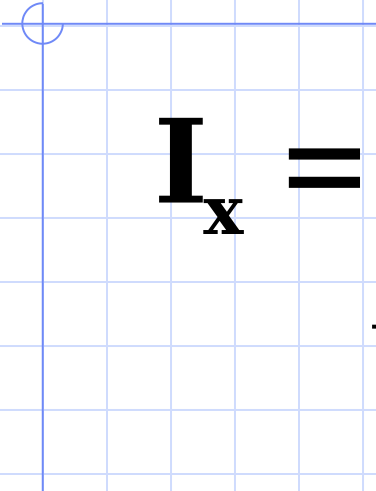


Figure 10.07(a)


$$I_x = \int_A y^2 dA = \int_A y^2 (2x) dy$$

$$= \int_{-a}^a y^2 \left( 2 \left( \sqrt{a^2 - x^2} \right) \right) dy = \frac{\pi a^4}{4}$$

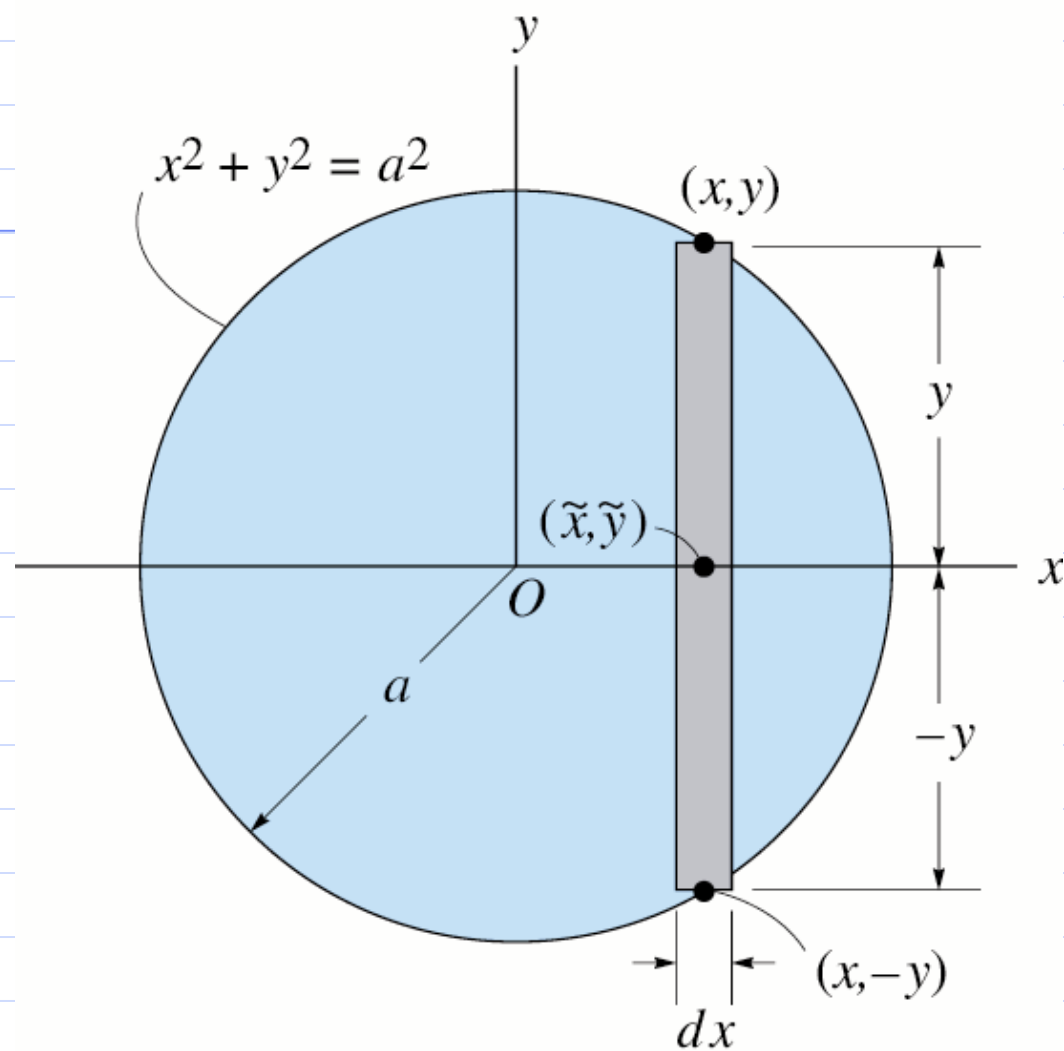


Figure 10.07(b)

$$\bullet \quad dI_x = \frac{1}{12} dx (2y)^3 = y^3 dx$$

$$I_x = \int_{-a}^a \frac{2}{3} (a^2 - x^2)^{\frac{2}{3}} dx = \frac{\pi a^4}{4}$$

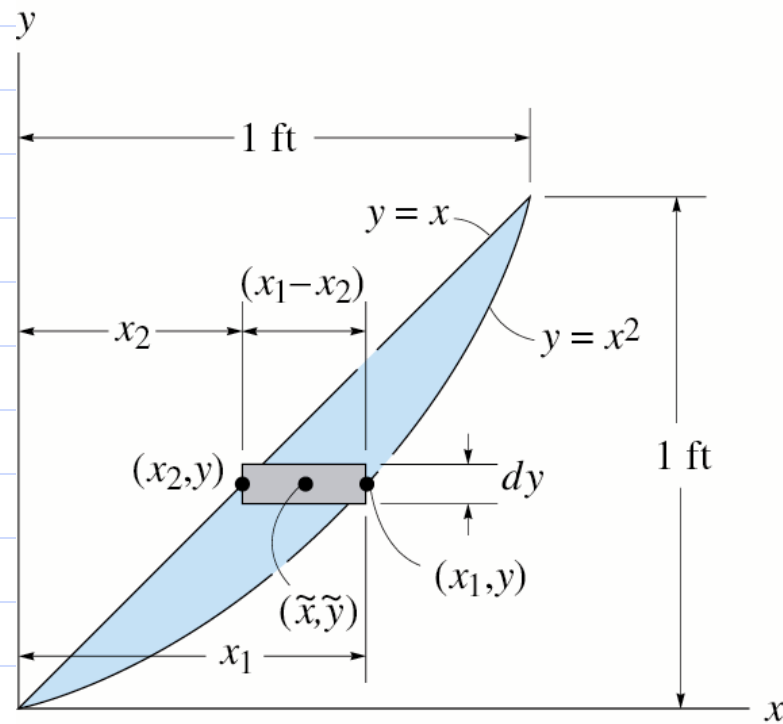


Figure 10.08(a)

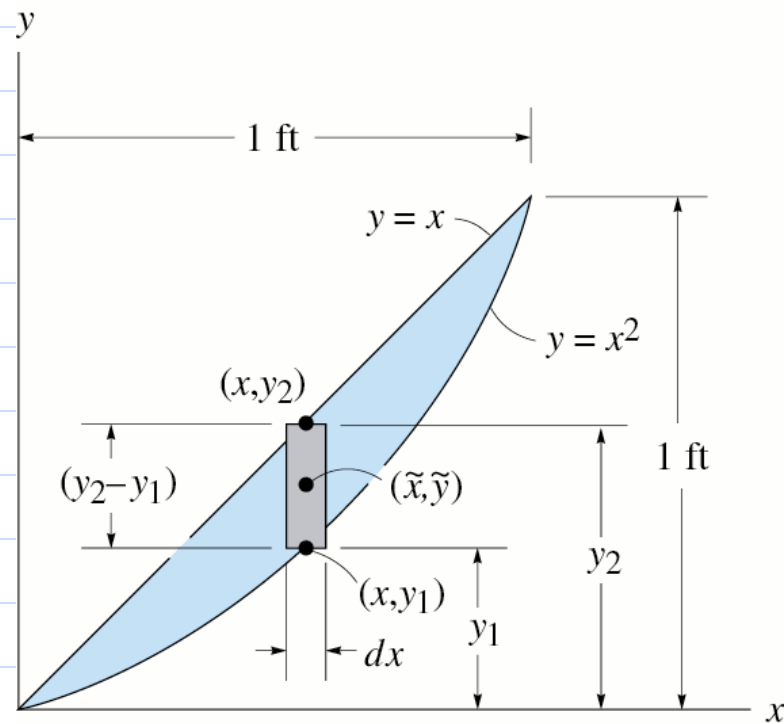


Figure 10.08(b)



# Composite Areas

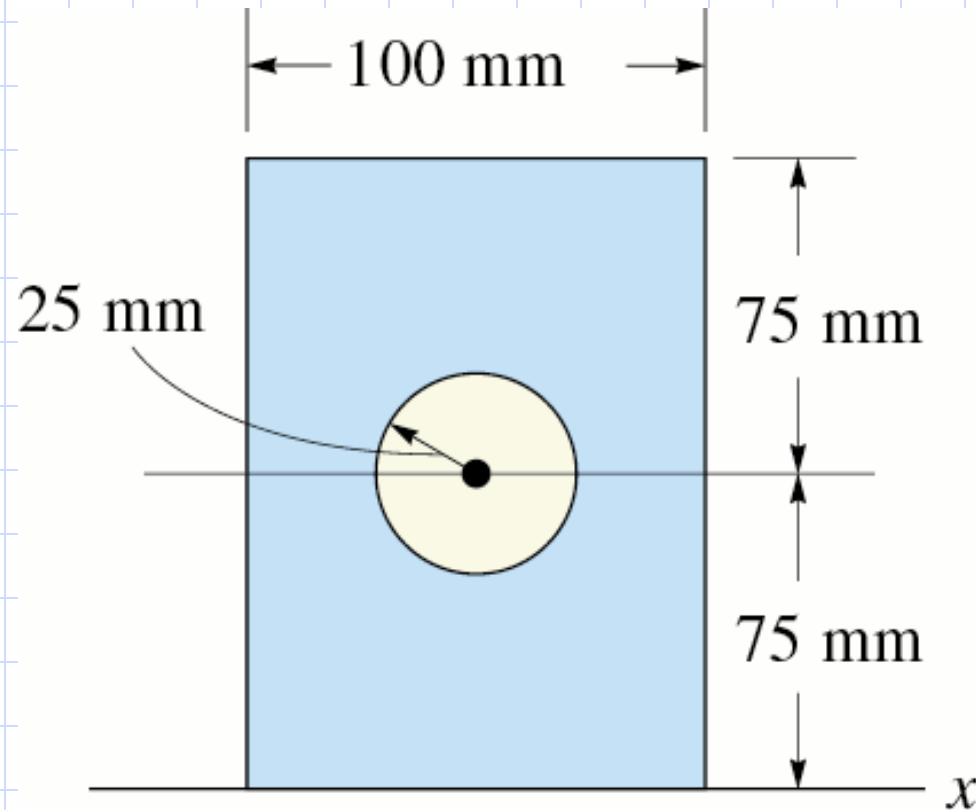


Figure 10.09(a)

# Composite Areas

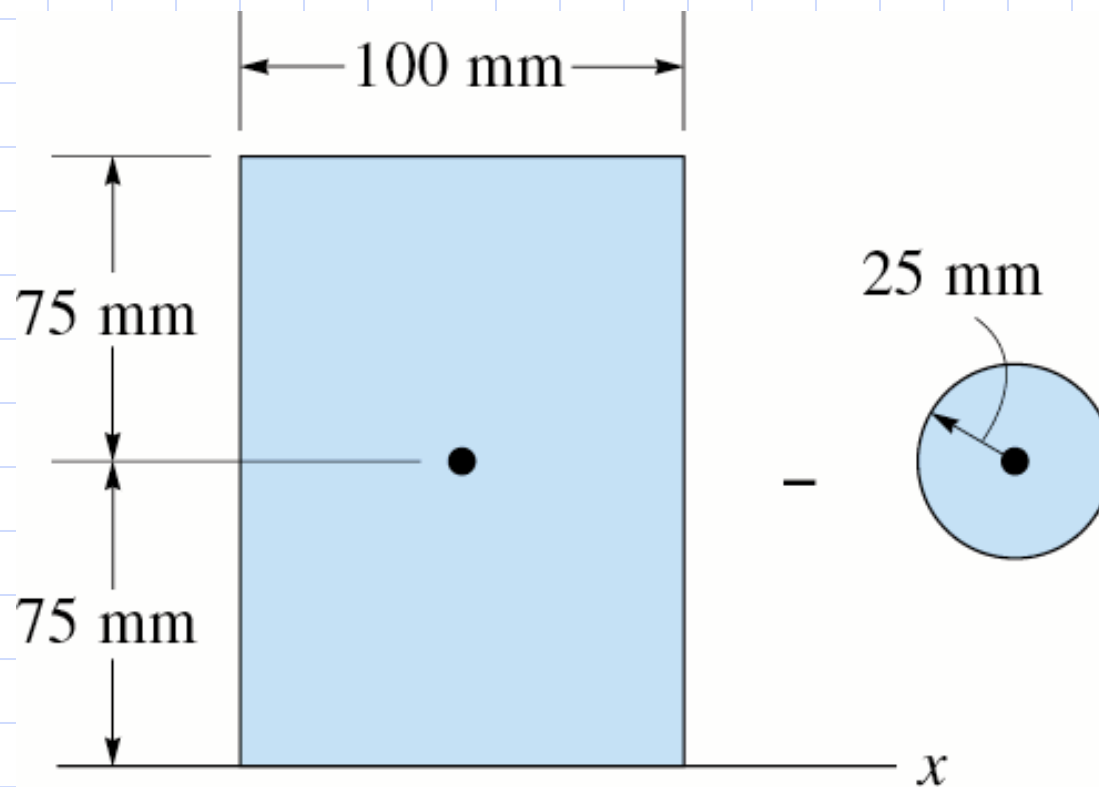
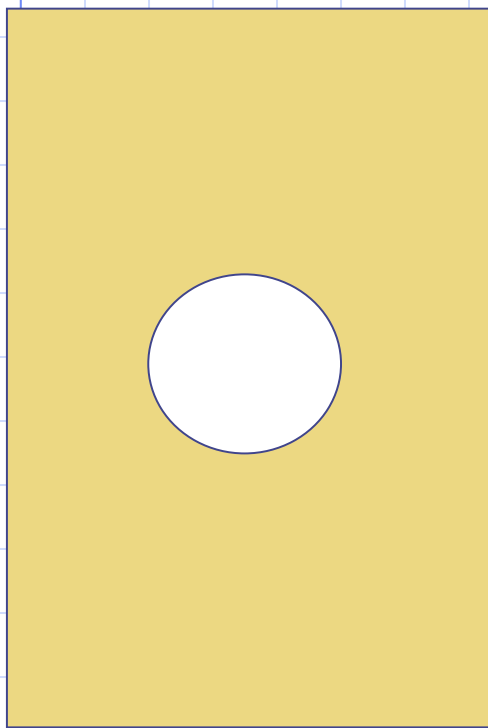
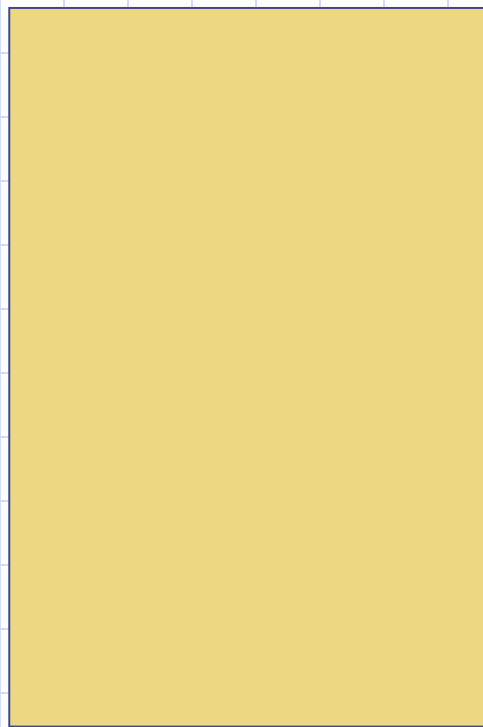


Figure 10.09(b)

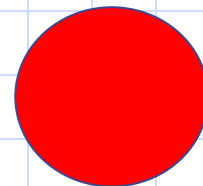
# Composite Areas

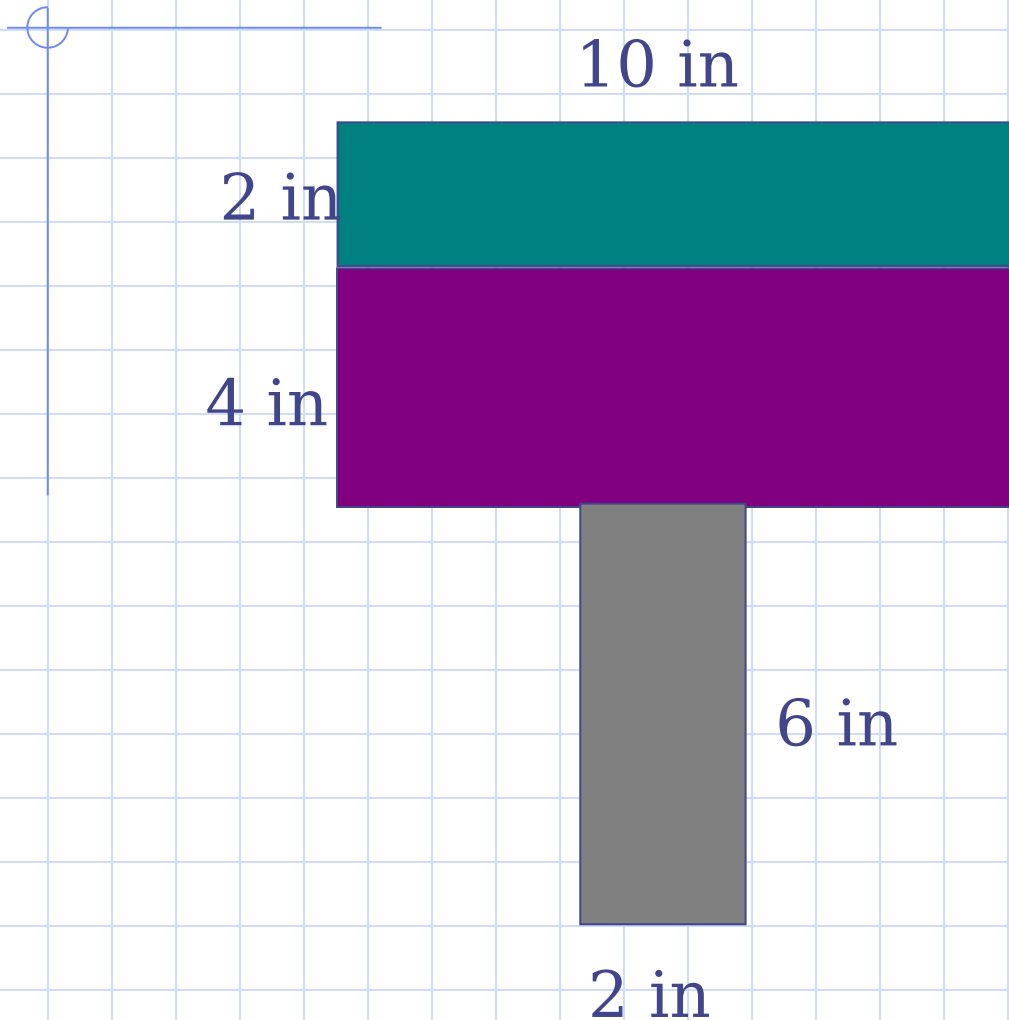


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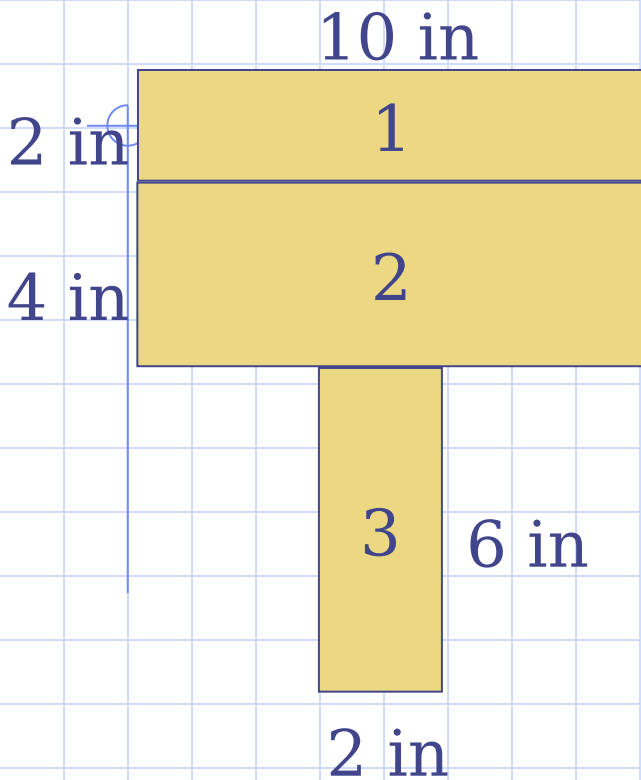


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**Locate  
centroid and  
moments of  
inertia  
about  
centroidal  
axes**



1	20	5	11	100	220
2	40	5	8	200	320
<u>3</u>	<u>12</u>	<u>5</u>	<u>3</u>	<u>60</u>	<u>36</u>
	72			360	576

$$\bar{x} = \frac{360}{72} = 5$$

$$\bar{y} = \frac{576}{72} = 8$$

$$I_x = I_{x1} + I_{x2} + I_{x3}$$

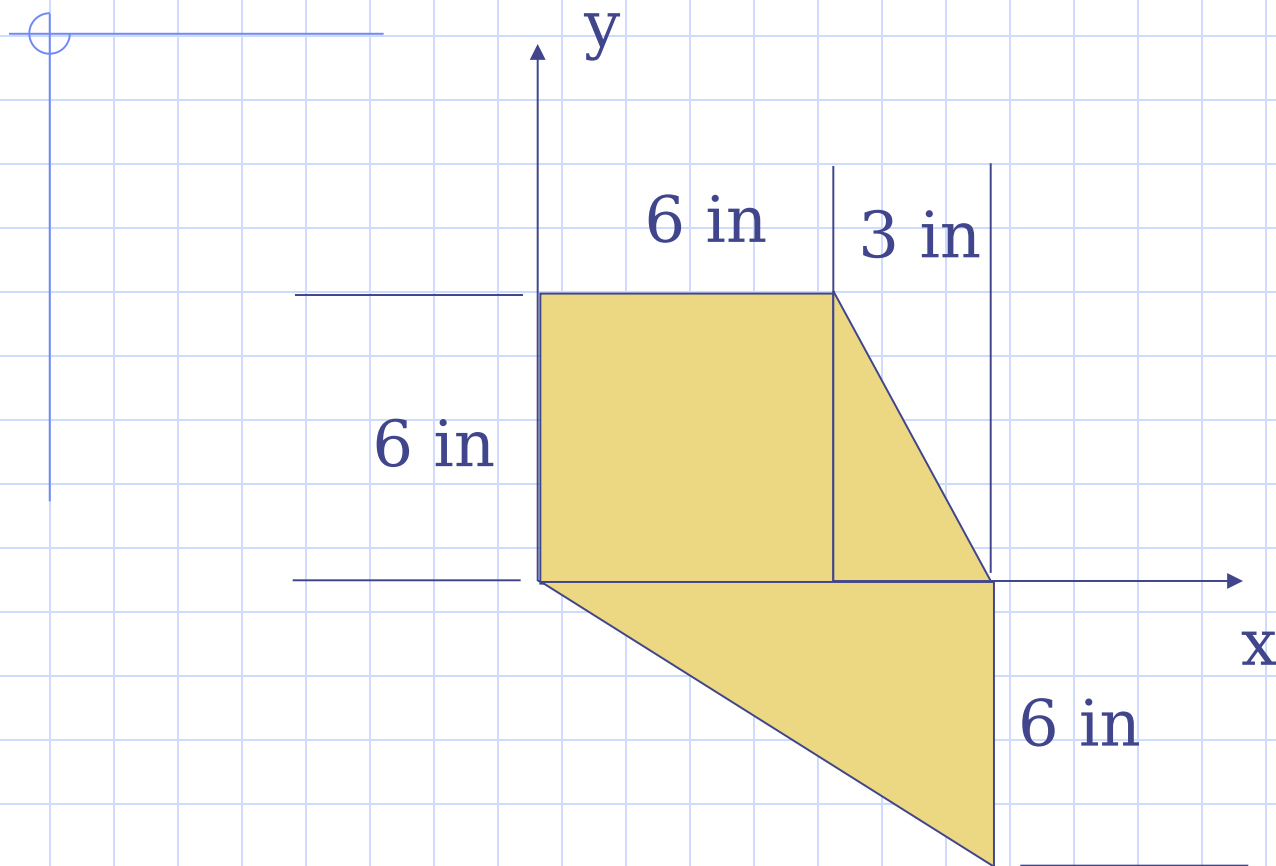
$$I_{x1} = \frac{1}{12} 10(2)^3 + 10(2)(11-8)^2 = 186\frac{2}{3}$$


$$I_{x2} = \frac{1}{12} 10(4)^3 + 10(4)(8-8)^2 = 53\frac{1}{3}$$

$$I_{x3} = \frac{1}{12} 2(6)^3 + 6(2)(8-3)^2 = 336$$

$$I_x = 186\frac{2}{3} + 53\frac{1}{3} + 336 = 576$$

Find  $I_x$  and  $I_y$

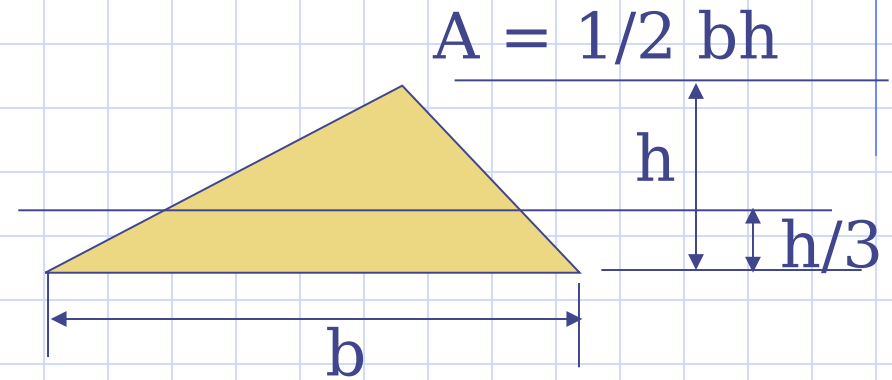
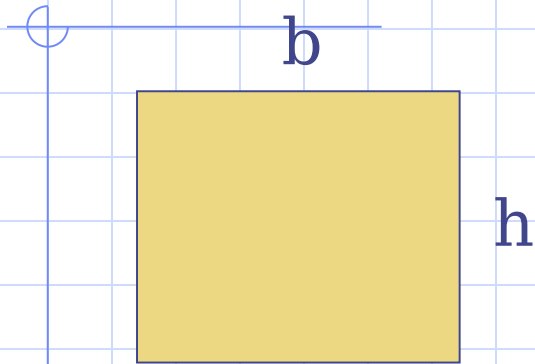




<b>Region</b>	<b>Area</b>	<b><math>I_x</math></b>	<b><math>I_y</math></b>	<b><math>d_x</math></b>	<b><math>d_y</math></b>
<b>1</b>	<b>36</b>	<b>108</b>	<b>108</b>	<b>3</b>	
<b>2</b>	<b>9</b>	<b>18</b>	<b>45</b>	<b>7</b>	
<b>3</b>	<b>27</b>	<b>54</b>	<b>121.5</b>	<b>6</b>	<b>2</b>



$$I_y = \left[ \frac{1}{12} (b) (h)^3 + (b) (h) \left( \frac{b}{2} \right)^2 \right] + \left[ \frac{1}{36} (b) (h)^3 + \frac{1}{2} (b) (h) (d)^2 \right] + \left[ \frac{1}{36} (b) (h)^3 + \frac{1}{2} (b) (h) (d)^2 \right]$$

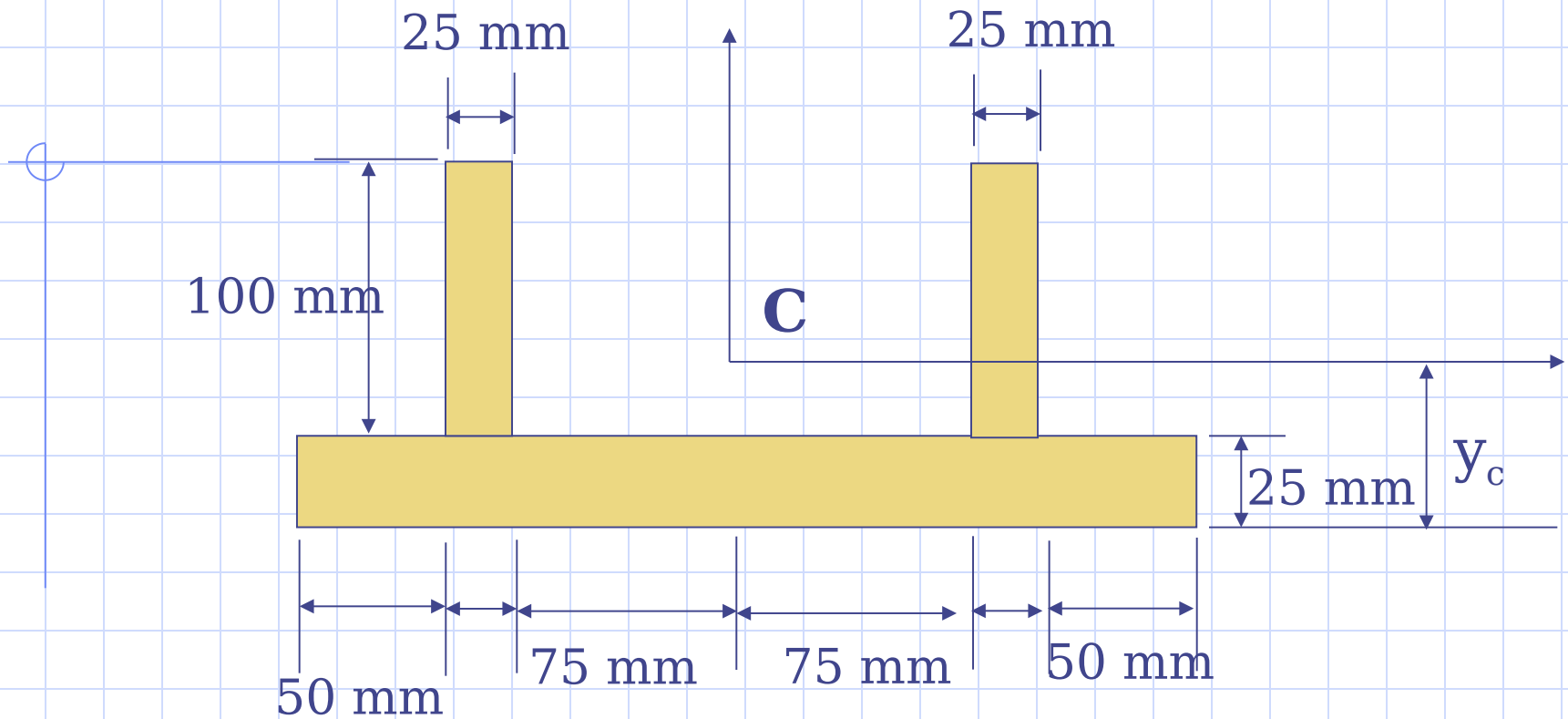


$$I_y = \left[ \frac{1}{12} (6) (6)^3 + (6) (6) (3)^2 \right] + \left[ \frac{1}{36} (6) (3)^3 + \frac{1}{2} (6) (3) (7)^2 \right] + \left[ \frac{1}{36} (6) (9)^3 + \frac{1}{2} (6) (9) (9)^2 \right] =$$

**197 in<sup>4</sup>**

$$\begin{aligned}
 I_y = & \left[ \frac{1}{12} (6) (6)^3 + (6) (6) (3)^2 \right] \\
 & + \left[ \frac{1}{36} (6) (3)^3 + \frac{1}{2} (6) (3) (7)^2 \right] + \left[ \frac{1}{36} (6) (9)^3 + \frac{1}{2} (6) (9) (9)^2 \right] = \\
 & \mathbf{197 \text{ in}^4}
 \end{aligned}$$

$$\begin{aligned}
 I_x = & \left[ \frac{1}{12} (6) (6)^3 + (6) (6) (3)^2 \right] \\
 & + \left[ \frac{1}{36} (3) (6)^3 + \frac{1}{2} (6) (3) (2)^2 \right] + \left[ \frac{1}{36} (9) (6)^3 + \frac{1}{2} (6) (9) (2)^2 \right] = \\
 & \mathbf{630 \text{ in}^4}
 \end{aligned}$$



Determine  $I_x$  and  $I_y$  about the centroidal axes.

Region	Area	x	y	xA	yA
1	7500	0	12.5	0	93750
2	2500	-87.5	75	-218750	187500
3	2500	87.5	75	218750	187500
	12500			0	468750

$$y_c = \frac{468750}{12500} = 37.5$$

$$\begin{aligned}
 I_{\bar{x}} = & \left[ \frac{1}{12} (300 \cdot 25^3 + (300 \cdot 25)(375 - 125)^2) \right] \\
 & + \left[ \frac{1}{12} (25(100^3 + (100 \cdot 25)(375 - 75)^2) \right] \\
 & + \left[ \frac{1}{12} (25(100^3 + (100 \cdot 25)(375 - 75)^2) \right]
 \end{aligned}$$

$$I_{\bar{x}} = 163 \times 10^6 \text{ mm}^4$$

$$\begin{aligned}
 I_x = & \left[ \frac{1}{12} (25(300^3 + (300 \times 25)(0)^2) \right] \\
 & + \left[ \frac{1}{12} (100 \times 25^3 + (100 \times 25)(0 - 875)^2) \right] \\
 & + \left[ \frac{1}{12} (100 \times 25^3 + (100 \times 25)(875 - 0)^2) \right]
 \end{aligned}$$

$$I_x = 948 \times 10^6 \text{ mm}^4$$

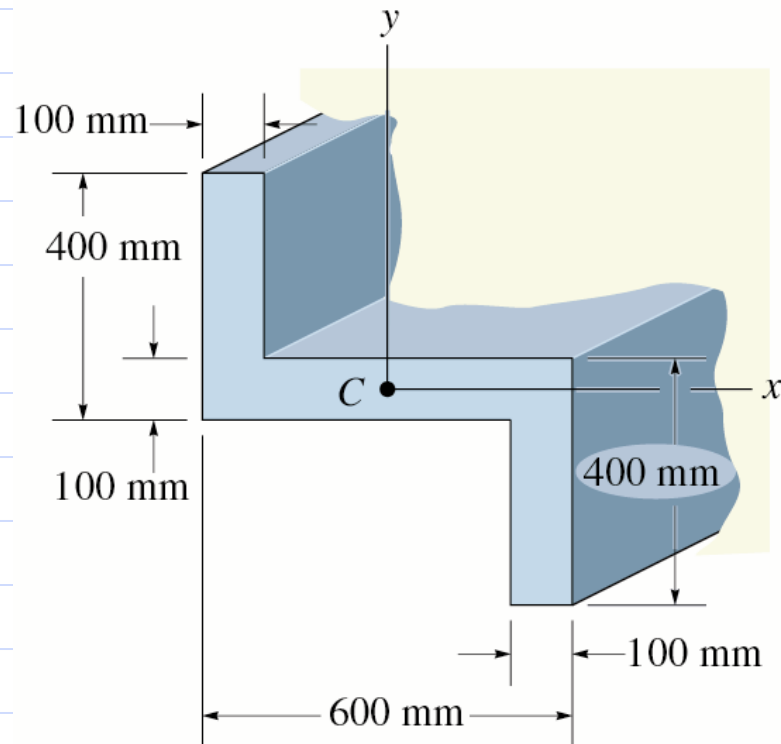


Figure 10.10(a)

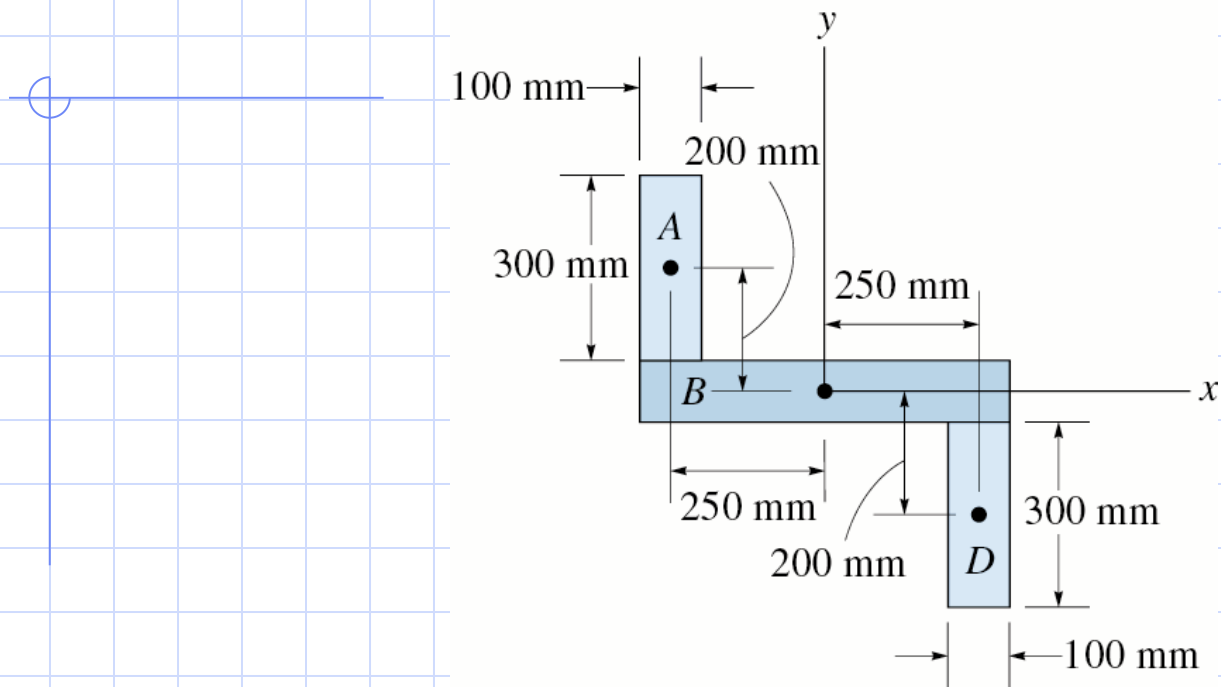


Figure 10.10(b)



# Product of Inertia

$$I_{xy} = \int_A xy dA$$

# Product of Inertia

$$I_{xy} = 0$$

***If either  $x$  or  $y$  is a line of symmetry.***

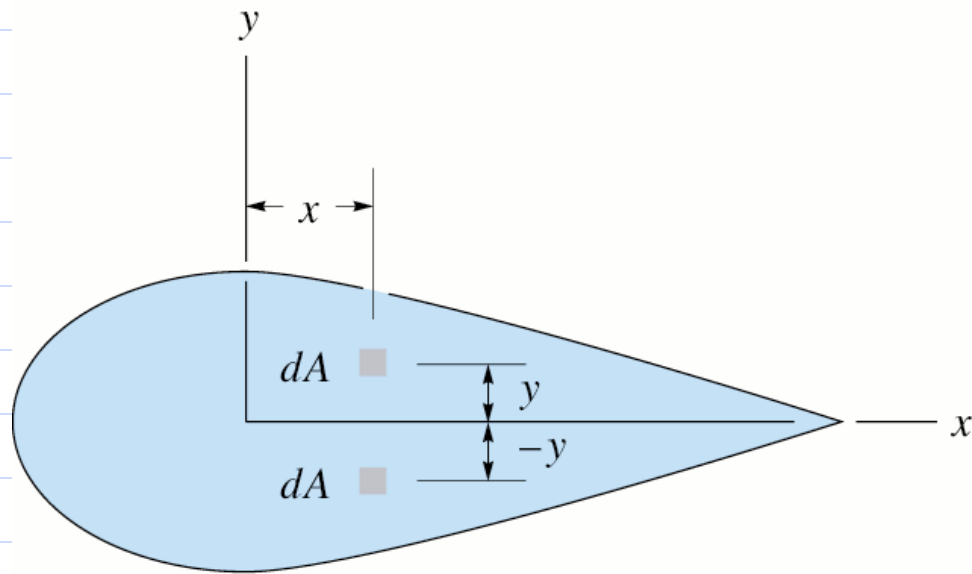


Figure 10.12

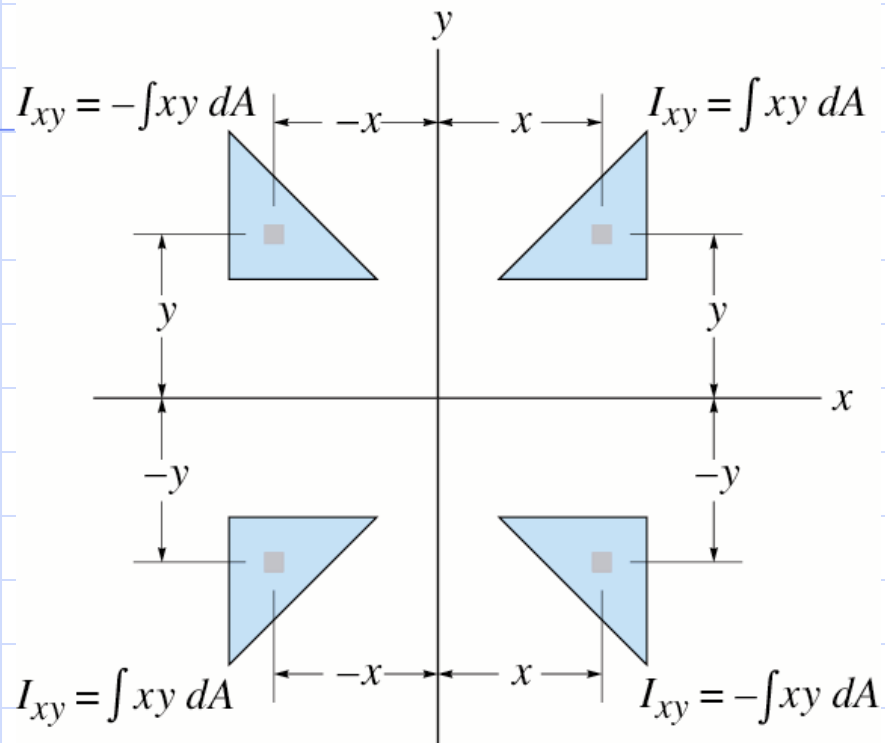


Figure 10.13

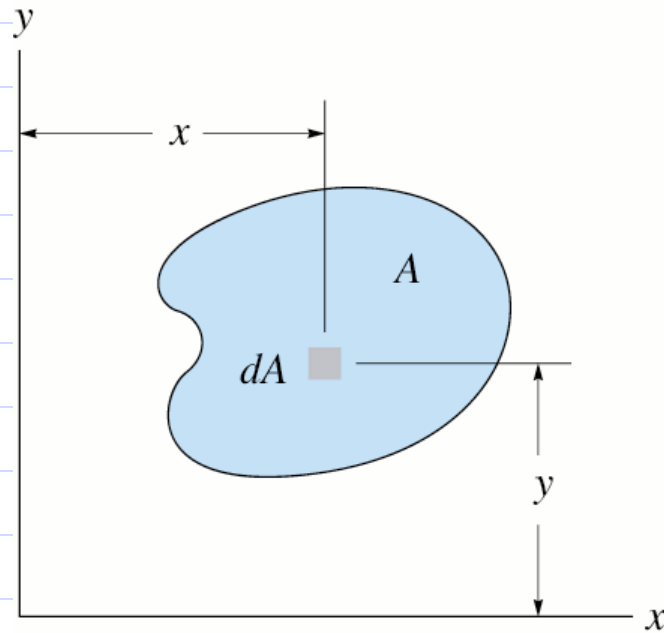


Figure 10.11

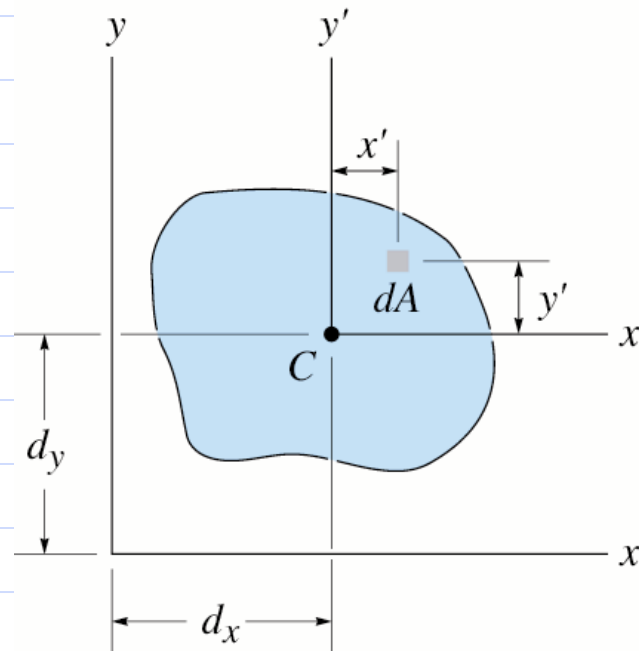


Figure 10.14

# Parallel Axis Theorem for Product of Inertia

$$\mathbf{I}_{xy} = \bar{\mathbf{I}}_{xy} + A \mathbf{d}_x \mathbf{d}_y$$

Algebraic signs of  $\mathbf{d}_x$  and  $\mathbf{d}_y$  are important!

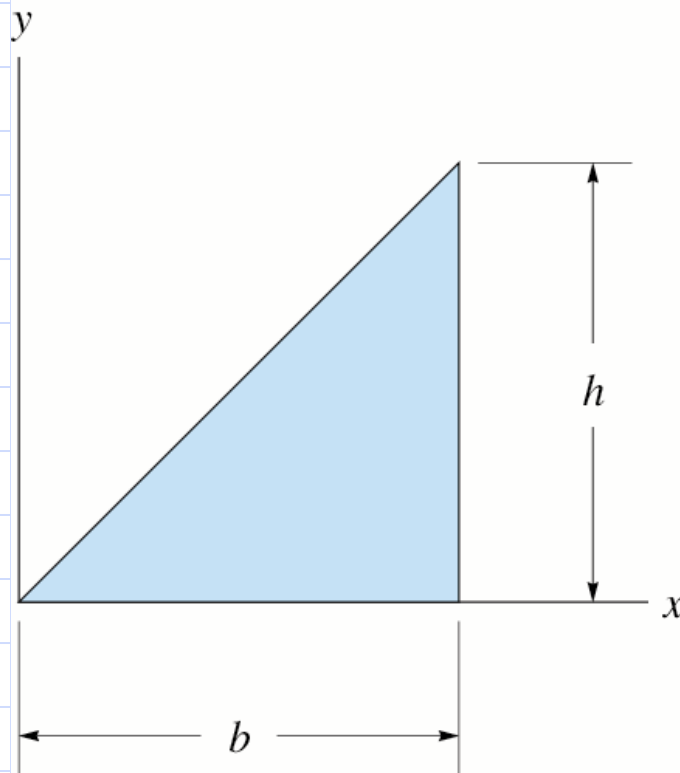


Figure 10.15(a)



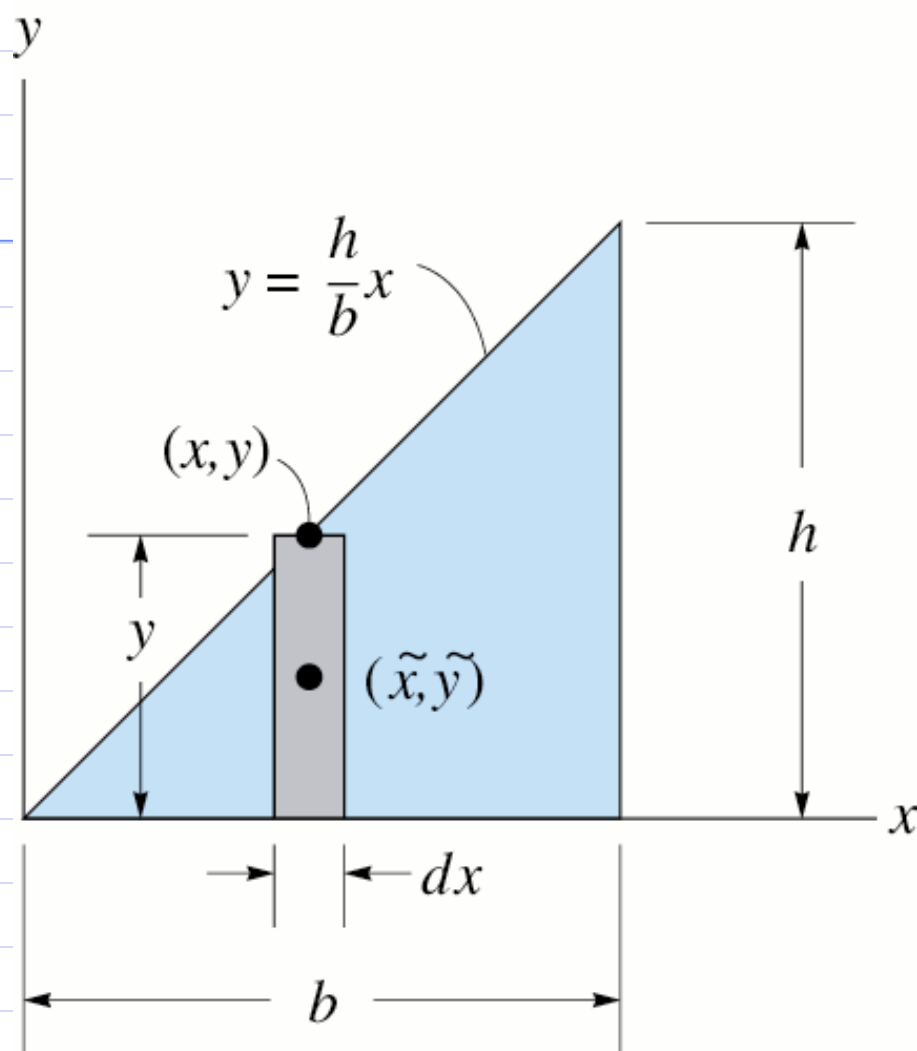


Figure 10.15(b)

$$dI_{xy} = d\bar{I}_{xy} + dA\bar{x}\bar{y}$$

$$dI_{xy} = 0 + (y dx) x \frac{y}{2}$$

$$dI_{xy} = 0 + \left( \frac{h}{b} x dx \right) x \frac{h}{2b} x$$

$$I_{xy} = \int_0^b \frac{h^2}{2b^2} x^3 dx$$

$$I_{xy} = \frac{b^2 h^2}{8}$$

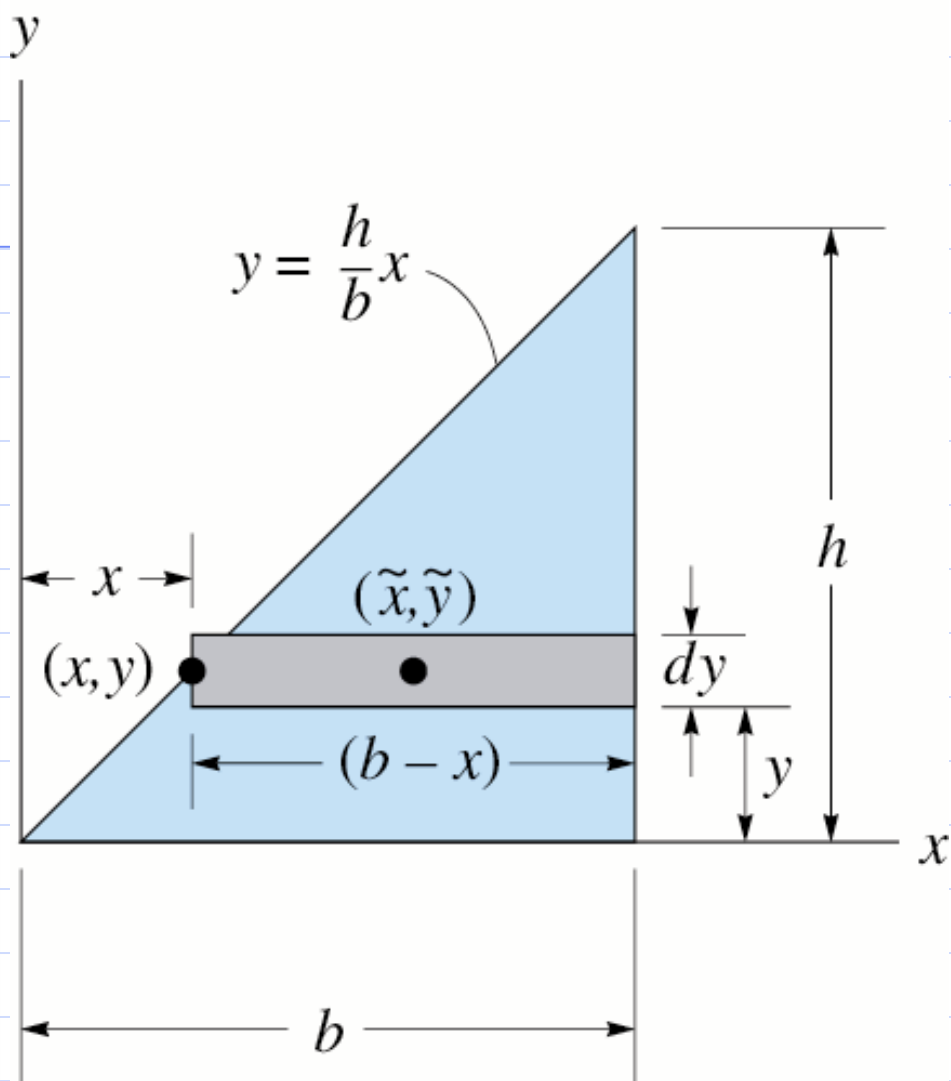


Figure 10.15(c)

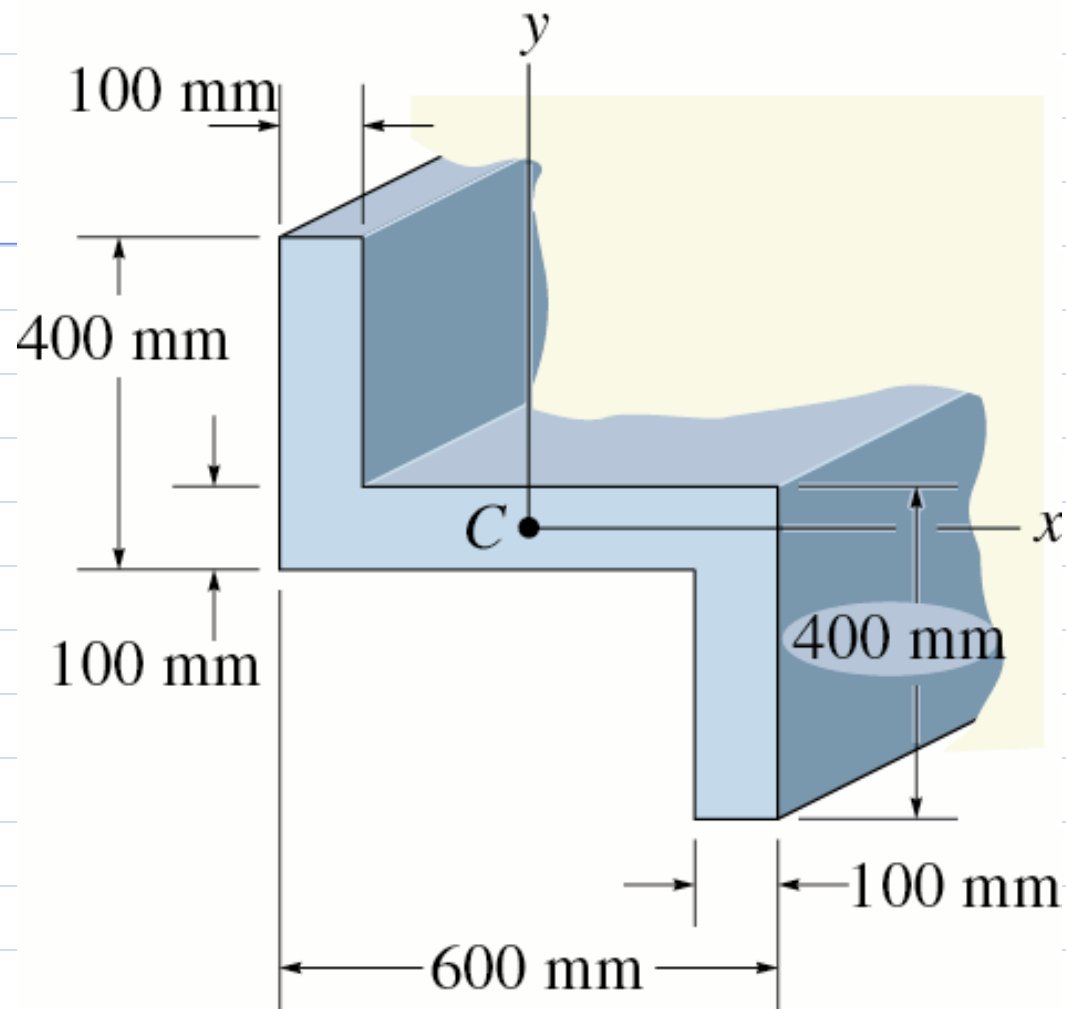
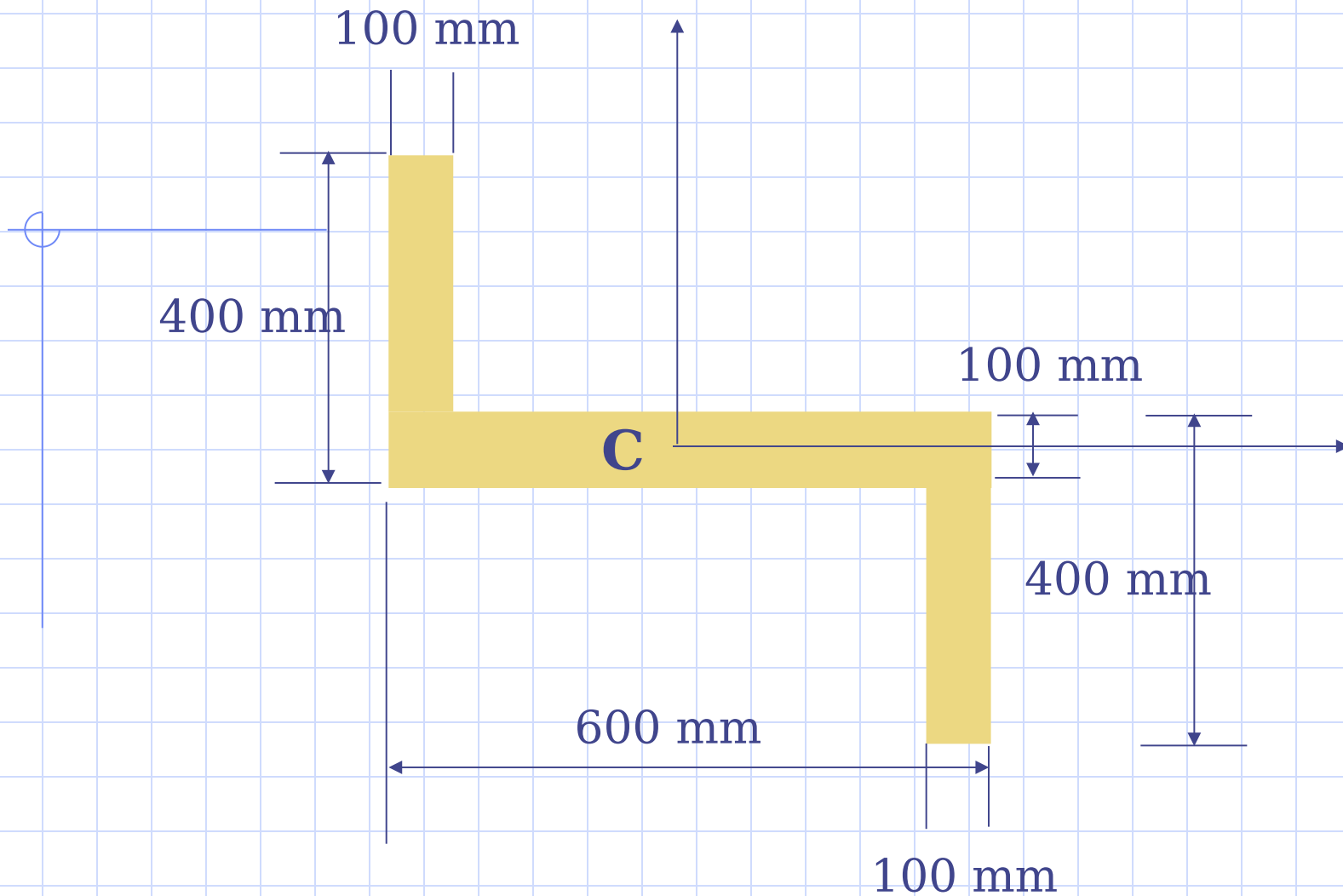
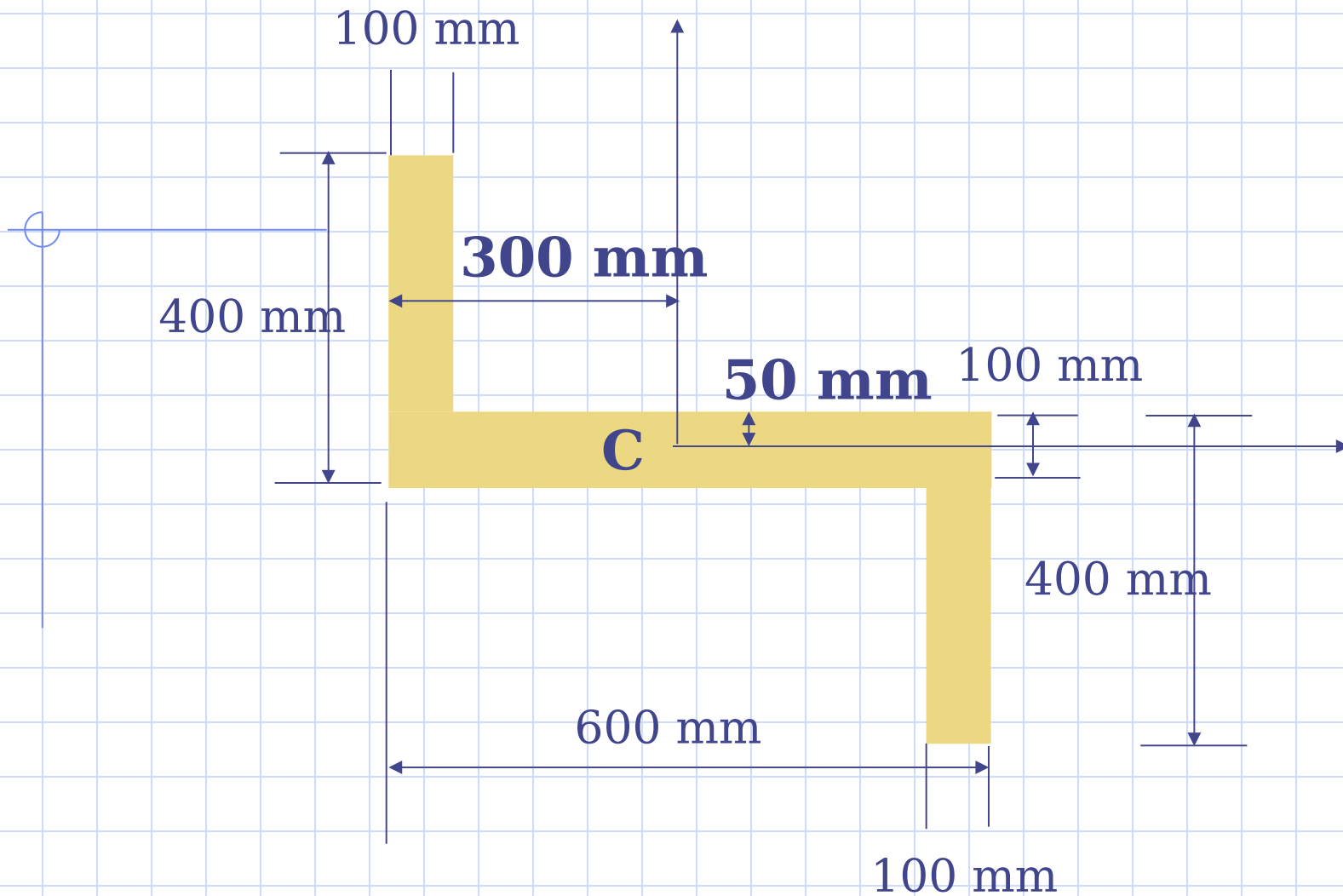


Figure 10.16(a)



Find the product of inertia about the centroidal axes.



Centroid due to symmetry.

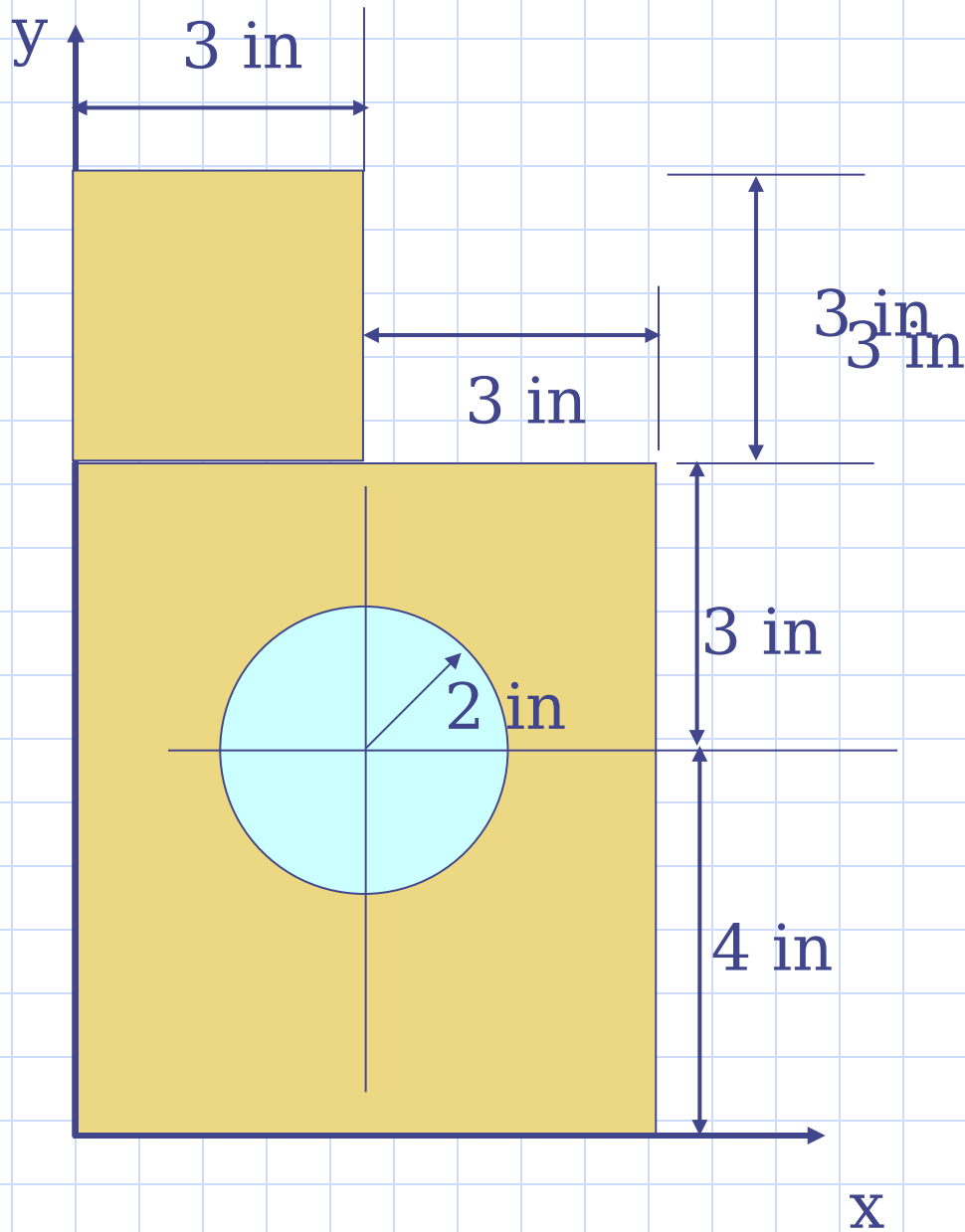


$$I_{xy} = [0 + (300 \times 1000 - 250 \times 200) \\ + [0 + (600 \times 1000)(0)] \\ + [0 + (300 \times 1000 \times 250 - 200)]$$

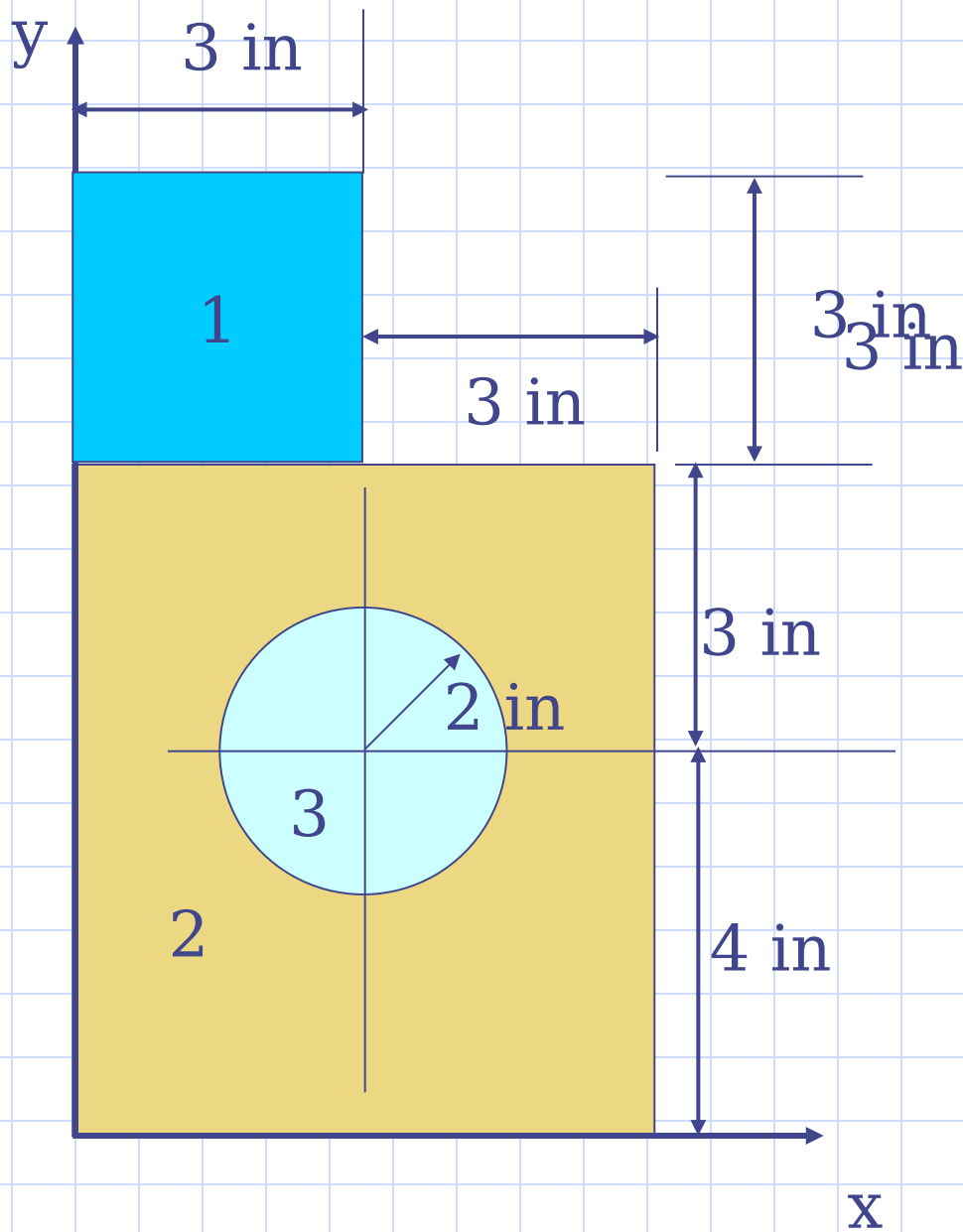
$$I_{xy} = -3.00 \times 10^9 \text{ mm}^4$$



Find  $I_x$ ,  $I_y$ , and  $I_{xy}$  about x-y axes and about centroidal axes



Identify sub-  
areas



# Find Centroid Location

Area #	Area	x	y	xA	yA
1	9	1.5	8.5	13.5	76.5
2	42	3	3.5	126	147
3	-12.566	3	4	-37.699	-50.265
	38.434			101.8	173.23
	x=	2.649			
	y=	4.507			

# Find Moment of Inertia about x-y Axes

$$I_x = \left( \frac{1}{12} 3(3)^3 + (3)(3)(8.5)^2 \right) + \left( \frac{1}{12} 6(7)^3 + (6)(7)(3.5)^2 \right) - \left( \frac{1}{4} \pi (2)^4 + (\pi)(2)^2(4)^2 \right) = 112972$$

$$I_y = \left( \frac{1}{12} 3(3)^3 + (3)(3)(1.5)^2 \right) + \left( \frac{1}{12} 7(6)^3 + (6)(7)(3)^2 \right) - \left( \frac{1}{4} \pi (2)^4 + (\pi)(2)^2(3)^2 \right) = 405363$$

# Find Moment of Inertia about x-y Axes

$$\begin{aligned} I_{xy} = & (0 + (3)(3)(8.5)(1.5)) \\ & + (0 + (6)(7)(3.5)(3)) \\ & - (0 + (\pi)(2)^2(4)(3)) \\ = & 404.9536 \end{aligned}$$

# nd Moment of Inertia about Centroidal A



$$I_x = \left( \frac{1}{12} 3(3)^3 + (3)(3)(8.5 - 4.50)^2 \right) + \left( \frac{1}{12} 6(7)^3 + (6)(7)(3.5 - 4.50)^2 \right) - \left( \frac{1}{4} \pi (2)^4 + (\pi)(2)^2(4 - 4.50)^2 \right) = 34854$$

$$I_y = \left( \frac{1}{12} 3(3)^3 + (3)(3)(1.5 - 2.649)^2 \right) + \left( \frac{1}{12} 7(6)^3 + (6)(7)(3 - 2.649)^2 \right) - \left( \frac{1}{4} \pi (2)^4 + (\pi)(2)^2(3 - 2.649)^2 \right) = 1356917$$

# Find Moment of Inertia about x-y Axes

$$\begin{aligned} I_{xy} = & (0 + (3)(3)(8.5 - 4.50)(1.5 - 2.64)) \\ & + (0 + (6)(7)(3.5 - 4.50)(3 - 2.64)) \\ & - (0 + (\pi)(2)^2(4 - 4.50)(3 - 2.64)) = -49486 \end{aligned}$$